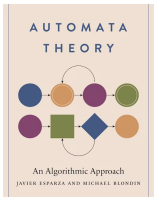
Review of [1]

Automata Theory: An Algorithmic Approach

Javier Esparza and **Michael Blondin**



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1 Overview

Consider the following problem: suppose we are given an infinite set U, and we want to represent subsets of U. The operations of interest on these subsets are the Boolean operations from set theory, such as intersection, union, and complementation. In addition, operations on relations—subsets of $U \times U$—such as projection and join (composition), must also be supported. The next table (taken from the book) summarizes these operations on sets and relations:

Table 1: Operations and tests for manipulation of sets and relations.

Operation on sets	**Returns**
`Complement`(X)	$U \setminus X$
`Intersection`(X, Y)	$X \cap Y$
`Union`(X, Y)	$X \cup Y$
Test on sets	**Returns**
`Member`(x, X)	**true** if $x \in X$, **false** otherwise
`Empty`(X)	**true** if $X = \emptyset$, **false** otherwise
`Universal`(X)	**true** if $X = U$, **false** otherwise
`Included`(X, Y)	**true** if $X \subseteq Y$, **false** otherwise
`Equal`(X, Y)	**true** if $X = Y$, **false** otherwise
Operation on relations	**Returns**
`Projection_1`(R)	$\pi_1(R) = \{x : \exists y \ (x, y) \in R\}$
`Projection_2`(R)	$\pi_2(R) = \{y : \exists x \ (x, y) \in R\}$
`Join`(R, S)	$R \circ S = \{(x, z) : \exists y \in X \ (x, y) \in R \wedge (y, z) \in S\}$
`Post`(X, R)	$post_R(X) = \{y \in U : \exists x \in X \ (x, y) \in R\}$
`Pre`(X, R)	$pre_R(X) = \{y \in U : \exists x \in X \ (y, x) \in R\}$

An infinite set has an uncountable number of subsets. The set of possible data structures, however, is countable, since each one is a finite object. As a consequence, there exist subsets that cannot be represented by any data structure. Thus, any data structure for manipulating infinite sets must strike a balance between expressiveness (which sets can be represented) and manipulability (which operations can be efficiently implemented).

The book presents a different perspective on the theory and applications of finite automata, both on finite and infinite words, highlighting their role as data structures capable of solving certain instances of this problem.

To give a taste of the technical content and style of the book, a few formal definitions are included throughout this review.

To begin the story of the book, we introduce its main character. Informally, a *semi-automaton* is an entity consisting of *states* and *transitions*. A distinguished nonempty subset of states is declared *initial*. Each transition has an *origin* and a *destination*, which are states, and a *label*, which is a symbol taken from some alphabet Σ—a nonempty and finite set of symbols. A *word* over Σ is a finite sequence of symbols taken from Σ, and a *language* over Σ is simply a subset of Σ^*, the set of all words over Σ. An *ω-word* over Σ is a function from $\mathbb{N}$ to Σ. The set of all such functions is denoted by Σ^ω, and an *ω-language* is a subset of Σ^ω.

The evolution of a semi-automaton is captured by the notion of a *run* (or a *computation*), which is simply an alternating sequence of states and transitions that begins in an initial state (and, when the run is finite, it ends in a state), such that if p, t, q are consecutive elements of the run, and p, q are states and t is a transition, then p is the origin and q is the destination of t. Each run c spells out a word, which is obtained by concatenating the labels of the transitions that make up c.

Thus, a semi-automaton can be defined as a tuple (Q, Σ, δ, Q_0), where

- Q is a finite set of states;
- Σ is a finite input alphabet;
- $\delta : Q \times \Sigma \to 2^Q$ is a transition function; and
- $\emptyset \neq Q_0 \subseteq Q$ is the set of *initial states*.

A semi-automaton is said to be *nondeterministic* in general, and *deterministic* whenever Q_0 is a singleton set and $\delta(q, a)$ is a singleton set for each $q \in Q$ and $a \in \Sigma$. In this case, both sets are identified with their unique element. These two notions play a fundamental role throughout the text.

Specializations of the notion of semi-automaton yield data structures capable of representing (possibly infinite) subsets of an infinite set.

2 Summary of Contents

The first part of the book, consisting of Chapters 1 to 9, is dedicated to finite automata over finite words. The second part, comprising Chapters 10 to 14, deals with ω-automata—automata designed to handle infinite words.

Chapter 1 introduces regular expressions and variants of finite automata, and shows that these formalisms constitute what the authors refer to as a *trinity*. Each provides a finite mechanism for representing languages, and they are equally expressive in the sense that they define the same class of languages. We now turn to define the main entities of the chapter.

The discussion begins with the usual suspects: alphabets, words, and languages. In addition to the standard set-theoretic operations on languages, the chapter introduces language concatenation and iteration (Kleene star). It then moves to regular expressions, which are syntactic forms used to generate languages. A *regular expression* over an alphabet Σ is defined by the following grammar:

$$r ::= \emptyset \mid \varepsilon \mid a \mid (r\, r) \mid (r + r) \mid r^*,$$

where $a \in \Sigma$. The language $\mathcal{L}(r)$ generated by a regular expression r is defined inductively as follows:

$$\begin{aligned}
\mathcal{L}(\emptyset) &= \emptyset \\
\mathcal{L}(\varepsilon) &= \{\varepsilon\} \\
\mathcal{L}(a) &= \{a\} \\
\mathcal{L}(r_1 + r_2) &= \mathcal{L}(r_1) \cup \mathcal{L}(r_2) \\
\mathcal{L}(r_1 r_2) &= \mathcal{L}(r_1) \cdot \mathcal{L}(r_2) \\
\mathcal{L}(r^*) &= \mathcal{L}(r)^*
\end{aligned}$$

A language over Σ is said to be *regular* if it is generated by a regular expression over Σ.

A *nondeterministic automaton* is a tuple $A = (Q, \Sigma, \delta, Q_0, F)$, where

- (Q, Σ, δ, Q_0) is a finite semi-automaton; and
- $F \subseteq Q$ is the set of *final* states.

The automaton A is said to be *deterministic* when the underlying finite semi-automaton is deterministic.

Another fundamental notion is that of acceptance, which allows one to associate a language with each finite automaton. A word w is *accepted* by A if there is a run that spells w and ends in a state inhabiting F. The *language accepted* by A is the set $\mathcal{L}(A)$ of all words accepted by A.

A graph-theoretical perspective on nondeterministic automata reveals that, for the purpose of language acceptance, only the states reachable from some initial state are relevant. Such an automaton is said to be in *normal form*.

The central result of the chapter establishes the equivalence of the three core formalisms. For all languages L over Σ:

(i) L is a regular language;

(ii) L is accepted by a nondeterministic finite automaton;

(iii) L is accepted by a deterministic finite automaton.

To prove this equivalence, the chapter introduces two intermediate models: NFA-ϵ, which are NFAs allowing ϵ-transitions (i.e., transitions that consume no input symbol), and NFA-reg, in which transitions may be labeled by regular expressions rather than single symbols.

The proof that (ii) implies (iii) employs the well-known subset construction. This construction transforms a nondeterministic automaton A with n states into a deterministic automaton D with 2^n states such that $\mathcal{L}(A) = \mathcal{L}(D)$. The chapter also presents an algorithm that constructs D directly in normal form, helping mitigate the exponential state blowup in practice.

Chapter 2 deals with the minimization of DFA and the reduction of the number of states of an NFA. These are critical tasks when viewing automata as data structures for representing languages. It shows that the uniqueness of the minimal DFA provides a canonical representation of a regular language. The chapter also presents a heuristic for reducing the size of nondeterministic finite automata (NFA) and proves that computing a minimal NFA is a PSPACE-complete problem. Moreover, it shows that there is no canonical minimal NFA.

For a regular language $L \subseteq \Sigma^*$, a DFA A that accepts L is said to be *minimal* if every DFA that recognizes L has at least as many states as A.

The canonical automaton of a language plays a fundamental role. Given a language $L \subseteq \Sigma^*$ and a word $w \in \Sigma^*$, the *residual* of L with respect to w is the language

$$L^w = \{u \in \Sigma^* \mid wu \in L\}.$$

The *canonical* automaton of L is the automaton

$$C_L = (Q_L, \Sigma, \delta_L, q_{0L}, F_L),$$

where:

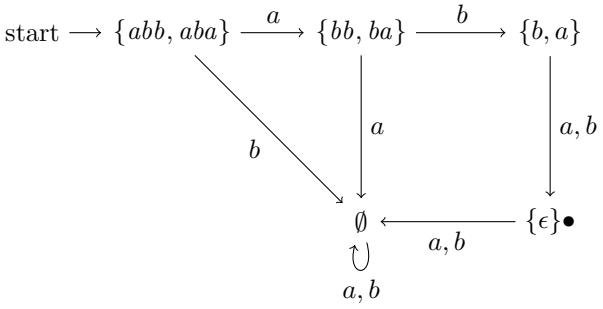

Figure 1: The figure illustrates the canonical automaton for $\{abb, aba\}$.

- $Q_L = \{L^w \mid w \in \Sigma^*\}$;
- $\delta_L(K, a) = K^a$, for each $K \in Q_L$ and $a \in \Sigma$;
- $q_{0L} = L$;
- $F_L = \{K \in Q_L \mid \varepsilon \in K\}$.

It is shown that for each regular language L the automaton C_L is the unique minimal DFA up to isomorphism that recognizes L (while the notion of isomorphism is not formally defined in the text, its meaning is made clear through the structure of the proof.).

The text presents Hopcroft's version of Moore's algorithm [3], which is a partition refinement algorithm whose main idea—to partition a block into two disjoint blocks—is also used to reduce the number of states of an NFA.

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton (DFA). *Moore's algorithm* consists of the following iterative procedure. Start with a partition $\mathcal{P}$ of Q such that every block $P \in \mathcal{P}$ satisfies either $P \subseteq F$ or $P \cap F = \emptyset$. That is, the initial partition is $\{F, Q \setminus F\}$. At each step, the algorithm checks whether the automaton obtained by collapsing each block $P \in \mathcal{P}$ into a single state yields a DFA. If so, the algorithm terminates. Otherwise, it identifies a block $P \in \mathcal{P}$ that is *non-deterministic* with respect to the current partition – meaning that there exist states $p, p' \in P$ and a symbol $a \in \Sigma$ such that $\delta(p, a)$ and $\delta(p', a)$ lie in different blocks of $\mathcal{P}$. In that case, the algorithm refines the partition by splitting P into two subsets:

$$P_1 = \{q \in P \mid \delta(q, a) = \delta(p, a)\} \quad \text{and} \quad P \setminus P_1.$$

The next iteration begins with the updated partition $(\mathcal{P} \setminus \{P\}) \cup \{P_1, P \setminus P_1\}$.

At this point, it is clear that finite automata provide a finite representation of a language. **Chapter 3** shows how to implement the set-theoretic operations listed in Table 1 under the following assumption: given a universe U, there exists an injection $s : U \to \Sigma^*$, called an *encoding*, such that the set $\{s(x) \mid x \in U\}$ forms a regular language over Σ. The chapter focuses on implementing operations on *regular subsets* of U – that is, subsets $X \subseteq U$ such that $\{s(x) \mid x \in X\}$ is regular – via corresponding constructions on finite automata. For instance, given two finite automata A_1 and A_2, it shows how to construct a new automaton that accepts $\mathcal{L}(A_1) \diamond \mathcal{L}(A_2)$, where $\diamond$ is a binary operation on languages. Methods are presented for computing the union, intersection, and complement of regular languages by manipulating the structure of deterministic or nondeterministic automata.

The implementation of binary operations involving two DFAs $A_i = (Q_i, \Sigma, \delta_i, q_{0i}, F_i)$ for $i \in \{1, 2\}$ can be achieved by taking the DFA induced by the set of states reachable from (q_{01}, q_{02}) of the *pairing* (or product) of the automata, i.e, the DFA $A = (Q_1 \times Q_2, \Sigma, \delta, (q_{01}, q_{02}), F)$, where $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$ for each $q_1, q_2 \in Q_1 \times Q_2$ and $a \in \Sigma$, and appropriately selecting the set F of final states. The text shows an algorithm that given two automata returns a product automaton that is in normal form. It also emphasizes

that this construction does not, in general, preserve minimality, and so the resulting product automaton is not necessarily minimal.

Beyond these operations, the chapter addresses the following decision problems related to regular languages:

- *Membership*: whether a given word belongs to the language accepted by a given finite automaton;
- *Emptiness*: whether the language accepted by a finite automaton is empty;
- *Universality*: whether a finite automaton accepts all words over the alphabet;
- *Equality*: whether two finite automata accept the same language;
- *Containment*: whether the language of one finite automaton is contained in that of another finite automata.

All these problems are shown to be decidable, and efficient algorithms are presented in the case of DFAs. While membership and emptiness remain tractable for NFAs, the chapter closes by proving that universality and language inclusion are both PSPACE-complete problems. However, for both universality and inclusion, a refined *subsumption-based* algorithm, which is variant of subset construction, is introduced to control the explosion of explored subsets in practice.

The development throughout the text provides a clear picture of the view of finite automata as a data structure with its emphasis on constructions and algorithms. The PSPACE-completeness proof showcases the limits of this paradigm when nondeterminism is involved.

Chapter 4, the first chapter devoted to applications, focuses on two instances of the *pattern-matching problem*. In the first, given a nonempty word $t \in \Sigma^*$ (the *text*) and a regular expression p (the *pattern*) over Σ, the goal is to find the length j of the shortest prefix u of t – if one exists – such that a word in $\mathcal{L}(p)$ is a suffix of u. The second instance is the special case where p is itself a word in Σ^*.

For the first variant, the chapter presents NFA and DFA-based algorithms. These work by constructing a finite automaton A that accepts the language $\mathcal{L}(\Sigma^* p)$ and checking whether there exists a prefix of t that belongs to $\mathcal{L}(\Sigma^* p)$ using A. The trade-offs between the NFA and DFA constructions are discussed, particularly the potential state explosion when converting the regular expression into a DFA.

Next, the chapter turns to the second instance, the *word case*, in which the pattern p is itself a word. This restriction makes the construction of a minimal DFA recognizing $\mathcal{L}(\Sigma^* p)$ straightforward. To improve the time complexity of the resulting algorithm, the notion of *lazy DFAs* (a restricted case of a two-way automata of Rabin and Scott [1]) is introduced. The chapter concludes with a novel derivation of the celebrated Knuth–Morris–Pratt algorithm.

Chapter 5 returns to Table 1 to show how operations on relations can be implemented. The conditions for an encoding are relaxed as follows. Given a universe U and an alphabet Σ that includes a padding symbol, say $\#$, a *partial encoding* is an encoding $s : U \to \Sigma^*$ (as in Chapter 3) such that for every $x \in U$, the last letter of $s(x)$ is different from $\#$. An *encoding* of U over Σ^* derived from s then maps each $x \in U$ to the (infinite) set $\{s(x)\#^n \mid n \geq 0\}$. In this approach, each object x is encoded as a (regular) language over Σ.

Given such an encoding of U over Σ^* and an NFA A, an element $x \in U$ is:

- *accepted* by A if A accepts all encodings of x, and
- *rejected* by A if A accepts none of them.

Moreover, A is said to recognize a subset $X \subseteq U$ if

$$\mathcal{L}(A) = \{w \mid \exists x \in X : w \text{ is an encoding of } x\}.$$

To handle relations, the chapter introduces the notion of a *transducer* over an alphabet Σ—that is, an NFA over the alphabet $\Sigma \times \Sigma$. Given words $u = a_1 a_2 \cdots a_n$ and $v = b_1 b_2 \cdots b_n$ over Σ, a transducer T

accepts the pair (u, v) if it accepts the word $(a_1, b_1)(a_2, b_2) \cdots (a_n, b_n)$. This definition motivates the use of the aforementioned encoding scheme in this context, since transducers, as defined, are length-preserving. The notion of acceptance must then be extended to transducers in a way analogous to the extension made for NFAs.

The chapter then nicely presents algorithms for efficiently implementing the relational operations over regular relations – subsets of $U \times U$ which can be recognized by transducers – listed in the table, including projection, join, post-image, and pre-image. The presentation contains a carefully chosen set of examples that illustrate the constructions in a clear way.

Now, we turn to **Chapter 6**. Chapter 3 presented implementations of the operations of union, intersection, and complementation for DFAs. However, even when these algorithms are applied to minimal DFAs, the resulting automata are not necessarily minimal—except in the case of complementation. For operations involving relations, the situation is even more critical, as determinism may not be preserved, particularly in the case of projection and join operations. This chapter presents implementations that produce minimal DFAs as output, with no need for an additional minimization step, under the assumption that the universe is finite. In this setting, all elements of the universe can be encoded using words of fixed length over some alphabet.

It is possible to view the universe of regular languages over Σ as an automaton, called the *master automaton*, introduced in Chapter 3, whose states are the regular languages, and where there is a transition from a regular language L_1 to a regular language L_2 labeled by $a \in \Sigma$ if and only if $L_2 = L_1^a$. Moreover, the set of final states is the set of all regular languages that contain the empty word. For a regular language L, the automaton induced by the set of states reachable from L, called an *L-induced* automaton, is the minimal automaton that recognizes L. With this, it is possible to obtain directly a minimal automaton for representing any finite language (see Figure 2).

Let Σ^n denote the set of all words of length n, and consider a finite set $\mathcal{L}$ whose elements are subsets of Σ^n. A *multi-DFA* for $\mathcal{L}$ is simply the union of all L-induced automata, where the union is taken over each $L \in \mathcal{L}$. The chapter then proceeds with a detailed exposition of how to implement the operations listed in Table 1, both for sets and relations, in this special case of multi-DFAs. These constructions reuse the states of the master automaton, allowing efficient implementations of set and relation operations without duplicating structure unnecessarily.

The chapter closes by showing how binary decision diagrams can be uses as minimal automata of a certain kind.

Chapter 7 is the second one of the first part dedicated to applications, and turns to the problem of verification of safety properties of sequential and concurrent programs with bounded-range variables.

Consider programs whose semantics transform an entity, called a *configuration*, into another configuration. With this setup, we can, for instance, construct an automaton whose states are all reachable configurations, plus an initial state, say i. All of its states are final. There is a transition $c \xrightarrow{c'} c'$ from a reachable configuration c to a configuration c' if the semantics of the program allow the transformation of c into c'. Moreover, for each initial configuration c, there is a transition $i \xrightarrow{c} c$. We are interested in the case where the set of configurations, say C, is finite. An execution is thus a run of this automaton, and can be seen as a word over the alphabet C—that is, as a language over C.

To test whether the set of executions $E \subseteq C^*$ of a program satisfies a certain property, which could be given by a regular expression over C, we can construct the automaton representing the set of executions, as well as the automaton representing the subset P of C^* that satisfy the property, and then check whether their intersection, $E \cap P$, is empty. This is the idea behind automata-based verification.

Network automata, discussed next, constitute an alternative way of modeling a program. Roughly, the idea is to model the behavior of each component of a configuration using a nondeterministic finite automaton (NFA). Thus, a network automaton is a tuple $\mathcal{A} = \langle A_1, \ldots, A_n \rangle$ of NFAs, possibly over different alphabets.

The text presents an algorithm that constructs an equivalent NFA, the *asynchronous product*, from a given network automaton. This construction is related to pairing and in the worst case, grows exponentially with respect to the size of the network. It also provides a modification of this algorithm that computes an NFA – called the *system automaton* – which represents the set of all executions of a program modeled by

the network automaton.

Next, it shows how network automata can model concurrent programs. To this end, the text uses Lamport and Burns's mutual exclusion algorithm as an illustrative example.

The last section of the chapter is dedicated to the state explosion problem. As previously discussed, the verification task essentially reduces to deciding whether the intersection $\mathcal{L}(A) \cap \mathcal{L}(r)$ is empty, where A is an automaton that recognizes the set of program executions over the configuration space C, and r is a regular expression that generates the subset of C^* consisting of traces that violate the desired property. Notice that when the system is modeled as a network automaton, A corresponds to the asynchronous product of the individual components, and thus its state space may grow exponentially with the size of the network. The verification problem is shown to be PSPACE-complete. An algorithm is then presented to check whether a property is violated, which, in a sense, constructs the asynchronous product on the fly.

The chapter concludes with a discussion of compositional verification and symbolic state-space exploration, followed by a short treatment of safety and liveness properties.

In **Chapter 8**, the relationship between automata and logic is explored. Consider, for instance, the following declarative definition of a language: the set of words over a, b that contain an even number of as and an even number of bs. It is straightforward to construct a finite automaton that recognizes this language. Producing a regular expression that generates it, however, is less immediate. Of course, the equivalence between both models allows one to convert from one representation to the other. As this example illustrates, for certain languages, a declarative definition is more natural. This motivates the study of logical formalisms that enable languages to be defined declaratively.

The chapter begins by introducing a predicate logic over words, essentially a first-order logic in which atomic formulas express properties of word positions, and quantification ranges over those positions. It then presents a fundamental result showing that this logic is not expressive enough to define all regular languages. Specifically, it is shown that the language $\{a^{2^n} \mid n \geq 0\}$ is not definable in this logic.

Next, the chapter introduces monadic second-order logic (MSO) on words, which extends first-order logic with two key features:

1. variables that range over sets of positions, and

2. the ability to express that a given position belongs to such a set.

Thus, in addition to quantification over positions, one can also quantify over sets of positions. The chapter presents a proof of the Büchi–Elgot–Trakhtenbrot theorem, which states that a language is regular if and only if it is definable by a formula in monadic second-order logic on words. As in previous chapters, several examples are provided to illustrate the constructions used in the proof, and they are particularly effective in clarifying the key ideas.

Chapter 9 discusses Presburger arithmetic and constitutes the third chapter on applications. Presburger arithmetic (PA) is a first-order language used to express numerical properties involving addition and comparison. The language of PA includes the constants 0 and 1, and features a single binary function symbol, namely $+$, which is written in infix form as usual. It also includes a single binary predicate symbol, $\leq$, likewise written in infix form. This syntax allows one to write formulas such as $\exists x \ (x \leq y + 1 \wedge y \leq x + x)$.

Let φ be a formula in PA. The chapter shows how to construct a transducer A_φ such that the language recognized by A_φ corresponds to the set of natural numbers (or integers) that satisfy φ. In line with the book's algorithmic perspective, the construction is clearly described through a careful exposition of the underlying algorithms. The techniques from Chapter 5 make it possible to reduce this problem to that of constructing a transducer for atomic formulas of the form

$$a_1 x_1 + \cdots + a_n x_n \leq b,$$

for which the chapter provides an algorithm along with a proof of correctness. It also addresses the special case of equations and presents an algorithm to handle that case as well. As in earlier chapters, the presentation is supported by clear and instructive examples that significantly aid comprehension.

Chapter 10 marks the beginning of the second part of the book, which focuses on the study of automata over infinite words. In line with Chapter 1, it begins by introducing the notion of an ω-regular expression over an alphabet Σ, which are syntactic forms generated by the following grammar:

$$s ::= r^\omega \mid rs \mid s + s,$$

where r is a regular expression over Σ. The meaning of an ω-regular expression s (defined recursively) is a subset $\mathcal{L}_\omega(s)$ of Σ^ω. This leads to the notion of an ω-regular language: a subset L of Σ^ω for which there exists an ω-regular expression s such that $L = \mathcal{L}_\omega(s)$. With this definition in hand, the chapter begins the search for data structures (automata) capable of representing such languages. The notions introduced here are more subtle, and the quest not only for a *trinity* among the various models presented, but also for closure under Boolean operations, becomes more elaborate and intricate.

The main character of the chapter is an *ω-automaton*, defined as a pair $A = (S, \alpha)$, where $S = (Q, \Sigma, \delta)$ is a finite semi-automaton and $\alpha : 2^Q \to \{0, 1\}$ is an *acceptance condition*. This abstract formulation elegantly subsumes all types of ω-automata presented in the chapter.

Given an infinite run ρ of S, let $\inf(\rho)$ denote the set of states that occur infinitely often in ρ. The run ρ is said to be *accepting* if $\alpha(\inf(\rho)) = 1$, and a word $w \in \Sigma^\omega$ is *accepted* by A if there exists an accepting run ρ of S that spells w. The language *recognized* by A is defined as

$$\mathcal{L}_\omega(A) = \{w \in \Sigma^\omega \mid A \text{ accepts } w\}.$$

The chapter first introduces Büchi automata and proves constructively that ω-regular expressions and Büchi automata are equivalent models with respect to ω-regular languages. A (nondeterministic) *Büchi automaton* (NBA) is an ω-automaton (S, α) such that α is a *Büchi condition*, meaning that there exists a set $F \subseteq Q$ such that $\alpha(Q') = 1$ if and only if $Q' \cap F \neq \emptyset$ for all $Q' \subseteq Q$. If S is deterministic, then A is called a *deterministic Büchi automaton* (DBA).

The chapter then shows that NBAs and DBAs are not equally expressive, and so DBAs cannot recognize all ω-regular languages. This is a manifestation that the notion of a *trinity* becomes more subtle in the setting of infinite words.

The next model introduced is the *co-Büchi automaton*, defined by an acceptance condition $\alpha : 2^Q \to \{0, 1\}$, called a *co-Büchi condition*, such that there exists a subset $F \subseteq Q$ with $\alpha(Q') = 1$ if and only if $Q' \cap F = \emptyset$ for all $Q' \subseteq Q$. The chapter presents a determinization algorithm that, given a nondeterministic co-Büchi automaton (NCA) A, returns a deterministic co-Büchi automaton D such that $\mathcal{L}_\omega(A) = \mathcal{L}_\omega(D)$—that is, both ω-automata recognize the same ω-language. However, it is also shown that NCAs do not recognize all ω-regular languages— yet another manifestation of the subtlety involved in establishing a trinity for infinite-word automata.

The search for the trinity culminates with the introduction of Rabin automata, the next model presented in the text. A *Rabin automaton* (NRA) is an ω-automaton $A = (S, \alpha)$ such that α is a *Rabin acceptance condition*; that is, there exists a finite set $\mathcal{R} \subseteq 2^Q \times 2^Q$ such that for each $Q' \subseteq Q$, we have:

$$\alpha(Q') = 1 \iff \exists (F, G) \in \mathcal{R} \text{ such that } Q' \cap F \neq \emptyset \text{ and } Q' \cap G = \emptyset.$$

In other words, a run ρ is accepting if there exists a pair $(F, G) \in \mathcal{R}$ such that ρ visits states in F infinitely often and states in G only finitely often.

The book provides a constructive proof of the equivalence between NRAs and ω-regular expressions, but refrains from proving the equivalence between nondeterministic and deterministic Rabin automata (DRAs). The complexity of union, intersection, and complementation is analyzed for DRAs. It is shown that union can be implemented by pairing, whereas intersection cannot. Moreover, complementation requires a modification of the underlying semi-automaton. Thus, unlike the case of deterministic finite automata, not all Boolean operations can be implemented with ease.

The chapter then delves into Streett, Parity, and Muller automata. Table 2, extracted from the text, summarizes the results involving these models.

Table 2: Summary of results of automata on infinite words.

Automaton Type	Expr.	Det.	Union	Inters.	Comp.
NFA/DFA	Y	Y	Y	Y	Y
NBA/DBA (Büchi)	Y	N	Y	N	N
NCA/DCA (Co-Büchi)	N	Y	N	Y	N
NRA/DRA (Rabin)	Y	Y	Y	N	N
NSA/DSA (Streett)	Y	Y	N	Y	N
NPA/DPA (Parity)	Y	Y	N	N	Y
NMA/DMA (Muller)	Y	Y	Y	Y	Y

- **Expr.** Every ω-regular expression (for the row NFA/DFA, every regular expression) can be converted into an automaton of this type.
- **Det.** For every nondeterministic automaton of this type, there is an equivalent deterministic automaton of the same type.
- **Union.** Union of deterministic automata of this type can be implemented using the pairing construction.
- **Inters.** Intersection of deterministic automata of this type can be implemented using the pairing construction.
- **Comp.** Complementation of deterministic automata of this type can be implemented without changing the semi-automaton or the type of the acceptance condition.

The entries of the table are as follows: **N** (the property does not hold), **Y** (the property holds, but the underlying conversion or algorithm requires exponential time), and **Y** (the property holds and the underlying conversion or algorithm only requires polynomial time). In particular, **Y** indicates that the resulting automaton has polynomial size in the input.

Chapter 11 shows how to implement binary operations on sets under the assumption that objects are encoded as ω-words, and thus the set of encodings defines an ω-regular language, making automata on infinite words a suitable data structure for representing these sets. Table 2 indicates that deterministic Muller automata can be used to implement union, intersection, and complementation, although all three operations incur a worst-case exponential blow-up. The table also shows that Rabin, Streett, and parity automata exhibit the same behavior with respect to two out of these three operations.

The chapter then moves from deterministic to nondeterministic automata. To this end, it introduces *nondeterministic generalized Büchi automata* (NGAs), which differ from Büchi automata by having a set $\mathcal{G}$ of accepting sets. A run is considered *accepting* if it visits each set in $\mathcal{G}$ infinitely often. The chapter provides an algorithm that, given a GBA, constructs an equivalent NBA. Moreover, it presents solutions to the union and intersection problems involving NGAs. The union construction, not surprisingly, resembles the one for NFAs, whereas the intersection relies on a pairing technique, supported by an algorithm that produces an NGA in normal form.

The complementation problem, however, cannot be addressed using the approach employed for NFAs, since there exist NGAs for which no equivalent deterministic generalized Büchi automaton exists. While the use of Muller automata could solve this issue, the corresponding construction is rather complex. Instead, the chapter follows a different strategy: it first transforms the NGA into an NBA and then applies a complementation algorithm for NBAs, which is presented in full detail throughout the remainder of the chapter.

It is worth to point out the use of the notion of the directed acyclic graph of a ω-word w – a construction that captures all possible runs of the automaton on an input word. This notion, introduced in Chapter 10 in the proof of equivalence between NCAs and DCAs, is employed again in proving the correctness of the complementation algorithm. Given a semi-automaton (Q, Σ, δ, Q_0) and a word $w = w_0 w_1 w_2 \cdots \in \Sigma^\omega$, the *directed acyclic graph* of w is the graph (V, E), where V and E are the smallest sets satisfying:

- $\langle q, 0\rangle \in V$ for every $q \in Q_0$;
- if $\langle q, i\rangle \in V$ and $q' \in \delta(q, w_i)$, then $\langle q', i+1\rangle \in V$ and $\langle q, i\rangle \xrightarrow{w_i} \langle q', i+1\rangle \in E$.

Chapter 12 is dedicated to the emptiness problem for NGAs, since problems such as membership,

containment, and equivalence can all be reduced to it. For the purposes of the emptiness problem, transition labels are irrelevant, so δ is identified with the set

$$\{(q, q') \in Q \times Q \mid \exists a \in \Sigma \text{ such that } (q, a, q') \in \delta\}.$$

The following assumptions are made throughout the chapter regarding the primitive operations available to the algorithms. For an NGA A with generalized Büchi acceptance condition $\mathcal{G}$, the algorithms are allowed access to:

(i) the set Q_0 of initial states;

(ii) the set $\delta(q)$ of successors of each $q \in Q$;

(iii) the collection $\{F \in \mathcal{G} \mid q \in F\}$ for each $q \in Q$.

Note that, because of (ii), the algorithms are restricted to forward exploration only. Such algorithms are called *on-the-fly*. This assumption aligns with other algorithms that operate in a similar fashion. For instance, algorithms for computing the intersection of two NGAs also construct the intersection in the forward direction. Therefore, to solve the emptiness problem for the intersection of two NGAs, one can compose both algorithms in a way that avoids explicitly constructing the intersection automaton.

The solution to the emptiness problem is presented next, based on two classical graph traversal algorithms: depth-first search (DFS) and breadth-first search (BFS). The problem is first addressed for nondeterministic Büchi automata (NBAs), where it reduces to deciding whether at least one state in the Büchi acceptance set F belongs to a cycle.

Although naive algorithmic solutions exist for this problem, the text develops a nested depth-first search strategy, which is described in detail along with a small optimization. It is observed that the nested DFS algorithm cannot be extended to NGAs, and therefore a conversion from an NGA to an NBA is required.

The text also explores a solution based on the notion of strongly connected components (SCCs) of a directed graph. A strongly connected component is a maximal subset $S \subseteq Q$ such that, for every pair of states $s, t \in S$, state t is reachable from s. It is well known that the directed graph obtained by collapsing each strongly connected component into a single node is acyclic. The text presents an algorithm, based on Tarjan's algorithm for computing SCCs, to decide the emptiness problem, which takes $O(|Q| + |\delta|)$ time. The advantage of this algorithm is that it can be easily extended to deal with NGAs.

The chapter closes with a description of breadth-first search algorithms for the emptiness problem in the special case of NBAs. It discusses the Emerson-Lei algorithm for NBAs and its generalization to NGAs.

Chapter 13 is devoted to the verification of liveness properties of programs, while **Chapter 14** addresses monadic second-order logic (MSO) over ω-words. The latter presents a construction of a Büchi automaton that recognizes the set of ω-words satisfying a given MSO formula. These two chapters were not read with the care they deserve and thus are not reviewed in detail here.

3 Opinion

The book presents a consistent and coherent account of finite automata operating on both finite and infinite words. It brings to light a new perspective – finite automata as data structures – which reflects the influence of key research areas in computer science, particularly verification, where algorithmic and data-structural concerns are central.

The book offers a clear and accessible exposition of its topics. The chapters are carefully organized, and each subject is developed in a steady and logical progression. Particular care is devoted to the choice of examples, which are well-crafted to illustrate the key ideas. Algorithms are presented in a style that is comfortable for computer scientists, although I personally favor a more functional approach that avoids side effects. A particularly strong point of the book is the inclusion of exercises along with detailed solutions. They complement and enhance the value of the text. Finally, the book includes previously unpublished

material and reformulations of known results, and delivers what it promises – a distinct perspective on automata theory – making it a valuable contribution to the literature.

Textbook presentations of topics related to ω-automata are scarce. For instance, the book by Khoussainov and Nerode [2], which is not cited in the bibliography, offers a presentation of Büchi, Muller, and Rabin automata with a distinct flavour that complements the present work. Connections to logic are also explored in Schneider [4] and Straubing [5], both likewise absent from the book's references. Schneider's text, in particular, provides a comprehensive and detailed treatment of verification. Finally, I would like to mention the book by Sakarovitch [3], which, in its first part, presents a beautiful account of classical automata theory (on finite words), and is also not cited in the bibliography.

As for prerequisites, a solid background in discrete mathematics is essential. Moreover, the book assumes what is often referred to – though never defined – "mathematical maturity", which includes comfort with formal definitions, inductive proofs, and abstract reasoning. While not strictly necessary, a familiarity with basic notions of automata theory and grammars would certainly be beneficial.

The book can be used at the undergraduate level, possibly as a second course in automata theory, or even as a first course, provided the students already have a solid mathematical and computer science background. Naturally, it also can be used as a textbook for an introductory graduate-level course.

References

[1] Hopcroft, J. and Ullman, J. *Introduction to Automata Theory, Languages, and Computation.* Addison-Wesley Publishing Company, First Edition, Reading (MA), 1979.

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[3] Jacques Sakarovitch. *Elements of Automata Theory.* Cambridge University Press, 2009.

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[5] Howard Straubing. *Finite Automata, Formal Logic, and Circuit Complexity.* Birkhäuser, 1994.