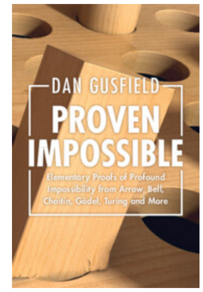
Review of [1]



Proven Impossible:
Elementary Proofs of Profound Impossibility
from Arrow, Bell, Chaitin, Gödel, Turing, and More
by **Dan Gusfield**

Cambridge University Press, 2024
$79.99 Hardcover, $34.99 Paperback, eBook, 270 pages

Review by **William Gasarch** (gasarch@umd.edu)

1 That's Impossible!

Some people outside of math think that statements like *You cannot write a program for HALT* are defeatist and pessimistic. They do not realize that these are rigorous theorems, and that it is good to know what you can't do, so you can modify your goals.

So clearly the layperson needs a book that gives coherent explanations of problems that are impossible to solve. The current literature seems to be in three overlapping categories:

1. Books for the layperson that are too fluffy and don't really get to the point.

2. Books for the layperson that oversell, for example, claiming *Gödel's Incompleteness Theorem proves that humans are creative!* or *Quantum computing will solve world hunger!*

3. Technical articles for experts that are not helpful to the layperson, even if they give (allegedly) simpler proofs of known theorems.

So this is a book for the layperson. How would it be for readers of this column? I suspect that $\frac{2}{3}$ of the people reading this review will enjoy $\frac{2}{3}$ of the book.

The chapters of the book do not quite correspond to theorems on impossibility, since some such theorems have two chapters about them or relate to other chapters. Hence I review the book not chapter by chapter but impossibility topic by impossibility topic.

2 Bell's Theorem

When quantum mechanics was first studied, the question *Can we model this using classical physics?* arose. This question would seem hard to formalize. Nevertheless, Bell's Theorem does just that: classical physics is formalized, and it is shown that quantum mechanics cannot be so described.

The book's explanation of Bell's Theorem is excellent. Amazingly it does not require knowing any quantum mechanics. The layperson will benefit; however, I suspect that my readers who don't live and breath quantum mechanics (that is, most of them) will also benefit from this chapter.

I note for future reference that Bell's Theorem is a profound statement about how our universe works. I will later comment on the profundity of the other chapters; however, none will top Bell's Theorem.

There are two chapters on Bell's Theorem.

3 Arrow's (and Friends) Theorem

As the readers of this column probably know, Arrow's Theorem states that, assuming reasonable assumptions about how an election can be run, the only system that satisfies them is a dictatorship. The *and Friends* in the title of the chapter refers to Gibbard-Satterthwaite Theorem, a similar result which uses different (more natural, according to Gusfield) assumptions and has a simpler proof.

Since voting is a man-made phenomenon, the results here cannot be as profound as Bell's Theorem. Also, there could be more discussion of what real voting systems look like and how they seem to not lead to dictatorships.

That said, this chapter gives a clean exposition of both Arrow's Theorem and Gibbard-Satterthwaite Theorem. The results are interesting. There is some discussion of why the real world seems to manage with voting despite these theorems (the consistency assumption is suspect, though it looks fine to me).

There is one chapter on voting theorems.

4 Clustering

Clustering is a grouping of data in the form of a set of points in the plane (or in d-dimensions) that will (hopefully) lead to a meaningful statement (e.g., *Basketball players tend to be tall.*) or a useful statement (e.g., *If we have to split up a club into smaller clubs, here is a way to do it so that close friends are in the same club.*).

Kleinberg gave three axioms about what a good clustering algorithm should achieve, and then showed that no clustering algorithm could achieve all three. This is reminiscent of Arrow's Theorem. In fact, once again one of the troubling axioms is about consistency.

Since clustering is a man-made phenomenon, the results here cannot be as profound as Bell's Theorem. However, in this case (unlike the chapter on voting) there is a discussion of why this theorem does not seem to be a problem in the real world of clustering.

Unlike Arrow's Theorem, this result is likely new to the reader (at least it was new to me). The presentation is clear and has the advantage of the proof being simplified since Kleinberg.

There is one chapter on clustering theorems.

5 Gödel's and Chaitin's Incompleteness Theorems

The readers of this review likely know Gödel's Incompleteness Theorems. Even so, it is good to revisit it from time to time. In addition, Chaitin's Theorem (which we discuss below) is related.

The first chapter on Gödel's Theorems contains some simple Gödel-ish statements and a proof of a simpler variant of the first incompleteness theorem. The second chapter on Gödel's Theorems contains the following more technical versions; indeed, this is the most technical chapter in the book.

1. *Gödel's First Incompleteness Theorem.* I paraphrase Theorem 9.5.1 (p. 207):

   Suppose a formal system Π *contains the language of arithmetic* $\mathcal{L}_A$*, and that* Π *is sound. If a particular subset of Gödel numbers is expressible in* $\mathcal{L}_A$*, then there is a true sentence that can be written in* $\mathcal{L}_A$ *that cannot be derived in* Π*, so* Π *is incomplete.*

   I prefer the following simpler version: *In all axiomatic systems commonly used in mathematics there are statements in math that are true but not provable in that system.* However, the formulation above serves to remind us that we need to state things carefully.

2. *Gödel's Second Incompleteness Theorem.* I quote the book exactly (p. 215):

   Any rich enough formal system Π *that can derive (inside* Π*) a statement implying that it is consistent is in fact inconsistent.*

   I *really* prefer the following simpler version: *If a system can prove its own consistency, then that system is inconsistent.* However, the formulation above serves to remind us that we need to state things carefully.

Gödel's Incompleteness Theorems have not affected how non-logicians go about their business. Why? Because the statements that Gödel proved were true but not provable were not statements of interest. They were contrived for the sole point of being true but not provable. One could argue that the Second Incompleteness Theorem is less contrived: one would want to prove that (say) ZFC is consistent. Even so, that is a concern of logicians.

Chaitin's Theorem gives a possibly less contrived example. We state it informally.

Let $\mathcal{L}$ *be a programming language. There is a constant* $u_{\mathcal{L}}$ *such that for every string* x *the statement "The shortest program in* L *that prints* x *is of length* $\geq u_{\mathcal{L}}$*." is not provable in any consistent formal system.*

Informally Chaitin's Theorem says that some strings (actually an infinite number of strings) do not have short descriptions, *and* this cannot be proven in any formal systems. In other words, it says that proving that a string is complicated is complicated.

These theorems are profound statements about mathematics.

There are two chapters on Gödel's Incompleteness Theorems and one on Chaitin's Theorem.

6 Turing Undecidability

The readers of this review likely know that HALT is undecidable. Even so, it is good to revisit it from time to time. I'll point out two things in this chapter that are of interest both to the layperson and the readers of this review.

1. Its not just HALT. Rice's Theorem states that *all nontrivial* properties of programs are undecidable.

2. The undecidability of HALT can be used to prove Gödel's First Incompleteness Theorem.

The chapter does a good job of leisurely explaining what HALT is, showing that it's undecidable, and exploring some consequences of this. The chapter does not say how this affects actual programmers and what they do about it.

That HALT (and other problems) are undecidable is a profound statement about computation.

There is one chapter on HALT.

7 Opinion

This is a great book both for the layperson and for people who already know some of the contents (the readers of this review are likely in the second category). So perhaps buy one for yourself and one for your math-inclined great-niece.

That said, here are some comments that are... not quite negative, but need to be said.

1. The topics tackled are of two types:

   (a) Those that say something profound: Bell's Theorem, Gödel's Incompleteness Theorems, Chaitin's Theorem, HALT is undecidable.

   (b) Those that are about man-made phenomena and hence, by their nature, are just not that profound: Arrow's Theorem on voting, Kleinberg's Theorem on clustering.

   I found the profound chapters to be more interesting.

2. I would have preferred to see more on the question *Once it is known that X is impossible, what happens next?*

3. I note the following omissions. This is *not* a complaint, since if the author puts in everything that could be put in, he would have a 1000-page book (that is why calculus textbooks are so big).

   (a) The three problems of antiquity: constructions of trisecting the angle, doubling the cube, squaring the circle. These are all impossible. There is a great book on these for the layperson: *Tales of Impossibility: The 2000-Year Quest to Solve the Mathematical Problems of Antiquity* by David Richeson. It was reviewed in a SIGACT News Book Review column here:

   `https://mathcs.clarku.edu/~fgreen/SIGACTReviews/bookrev/53-1.pdf`

   (b) P vs. NP. There is a great book on this for the layperson: *The Golden Ticket: P, NP, and the Search for the Impossible* by Lance Fortnow. It was reviewed in a SIGACT News Book Review column here:

   `https://www.cs.umd.edu/~gasarch/bookrev/44-3.pdf`

   (c) Independence of the Continuum Hypothesis. I do not know of any account of this for the layperson. There could be a book with some history and context; however, it would be hard to do any real math. There is a book with a chapter on the history and context of this problem: *The Honor Class: Hilbert's Problems and Their Solvers* by Ben Yandell. It was reviewed in a SIGACT News Book Review column here:

   `https://www.cs.umd.edu/~gasarch/bookrev/44-4.pdf`

   (d) Unsolvability of the quintic. I know of two books for the layperson on this problem, though I have not read them: (1) *Evariste Galois 1811-1832* by Laura Rigatelli, and (2) *The Equation that Couldn't Be Solved* by Mario Livio.

I re-iterate that this is a great book both for the layperson and for people who know some of the material.