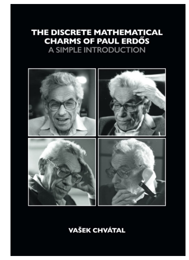
Review of [1]

The Discrete Mathematical Charms of Paul Erdős
A Simple Introduction
by Vašek Chvátal
Cambridge University Press, 2021
Paperback, 266 pages, $29.99



Review by
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1 Introduction

Paul Erdős (1913-1996) was an essential singularity in the world of twentieth-century mathematicians. The author Vašek Chvátal, who met Erdős as a young undergraduate in the mid '60s, co-wrote three papers with him and maintained a special friendship for the rest of Erdős' life. Donald Knuth wrote, "Vašek Chvátal was born to write this one-of-a-kind book. Readers cannot help be captivated by the evident love with which every page has been written. The human side of mathematics is intertwined beautifully with first-rate exposition of first-rate results."

Chvátal's book is based on his lecture notes for a graduate course *Discrete Mathematics of Paul Erdős*, which he taught at Concordia University in Montreal. But the book goes way beyond notes for a graduate course, being a reference on the many topics it covers, with full proofs and recent results. The book's eleven chapters include ones on Bertrand's Postulate, Discrete Geometry, Ramsey's Theorem, van der Waerden's Theorem, Extremal Graph Theory, the Chromatic Number, Hamilton Cycles, and others. The Summary below includes representative theorems from three of these chapters.

Each chapter ends with some non-mathematical reminiscences from the author about Erdős and his life, thoughtfully set apart by a slightly shaded background. The Summary below also includes several lesser-known things about Erdős or quotes by him.

2 Summary

To give a flavor of the mathematics in the book, I've chosen a key theorem from each of three chapters that serves as the beginning point for deeper results in them.

Bertrand's Postulate (chapter 1)

In 1845 Joseph Bertrand conjectured that for every positive integer n, there is at least one prime number p such that $n < p \leq 2n$. Chebyshev proved this theorem (the term "postulate" is a misnomer) in 1852, and in 1931 the precocious Erdős gave an "elegant elementary proof," according to Chvátal, that was published as his first article. Erdős' proof was a tour de force of estimates for binomial coefficients, Legendre's formula for the exponents of primes in the unique factorization of integers, and a clever approach to showing non-constructively that there must be a prime between

n and $2n$. Erdős used a similar proof technique in much later papers unrelated to number theory, which became known as "the probabilistic method." The chapter includes sketches of other proofs of Bertrand's Postulate, including a somewhat-similar one by Ramanujan, and recent results and problems concerning primes.

Van der Waerden's Theorem (chapter 6)

In 1927 Bartel van der Waerden published a proof of a conjecture that he heard from the Dutch mathematician Han Baudet: For each positive integer k, if all positive integers are colored red or white there is an arithmetic progression of k terms with the same color. Forty-four years later van der Waerden published the article "How the proof of Baudet's conjecture was found," describing an afternoon working on it with Emil Artin and Otto Schreier at the University of Hamburg. They first proved some special cases, and then tried strong induction on a multicolor version of the conjecture. Chvátal's book uses a generalization of the conjecture involving two variables k and l, then presents a rather tedious proof of it by double induction on k and l. Motivated by van der Waerden's Theorem, Erdős and Turán proposed several conjectures that eventually led to Szemerédi's Theorem and his award of the Abel Prize in 2012. The chapter concludes with a discussion of a theorem of Hales and Jewett (1961) from Ramsey Theory, which implies van der Waerden's Theorem.

The Friendship Theorem (chapter 8)

In 1966 Erdős and collaborators A. Rényi and V.T. Sós proved that if in a (finite) graph every pair of vertices has precisely one common neighbor, then some vertex is adjacent to all of the other vertices. This theorem became known as the Friendship Theorem, as it asserts that if in a group of people every pair has exactly one friend in common, then someone in the group is a politician, that is, he's a friend of everyone. The proof is broken into two main cases, when the graph is regular and when it's not. A regular graph is one where every vertex has the same degree, that is, the same number of neighbors. The easier case to prove is when the graph is not regular. The harder case of a regular graph is broken into two subcases, in one of which a big gun is used, Baer's Theorem: every polarity in a finite projective plane maps some point to a line that contains this point. The book does give an alternate proof which avoids Baer's Theorem, but it's several pages long and tedious.

The remainder of the chapter involves strongly regular graphs and uses tools of linear algebra applied to the adjacency matrix of a graph, such as the eigenvalues of a square matrix and the Principal Axis Theorem for real symmetric square matrices.

3 Reminiscences about Erdős and his life

Part of the appeal of the book is its inclusion of many anecdotes about Erdős, some well-known and others discovered by the author in his friendship with Erdős for over three decades. For example, Erdős was an itinerant mathematician known for his quote, "Another roof, another proof," who also had a maxim that "property is a nuisance." Chvátal notes that Erdős was the antithesis of a snob, and an anarchist in the noblest sense of the term. According to a declassified FBI file on Erdős that's excerpted in an appendix, he did not apply for U.S. citizenship as, "I am stateless by political conviction." Erdős' naiveté was illustrated in the early years of World War II when he, S. Kakutani, and A. Stone were briefly arrested on Long Island for taking photos against a background of what turned out to be a secret radar station! The book includes a photo from the *New York Daily News* the next day with the headline "3 ALIENS NABBED AT SHORT-WAVE

STATION."

Not all quotes that are often attributed to Erdős are valid. In a footnote Chvátal says that the famous saying "A mathematician is a machine for turning coffee into theorems" was actually due to Erdős' friend A. Rényi. But the stories of Erdős' heavy use of Benzedrine and Ritalin to sustain 19-hour workdays and sharpen his concentration were true, particularly after his mother died in 1971. In 1979 mathematician Ron Graham, Erdős' good friend and part-time caretaker, bet him $500 that he couldn't stay off drugs for a month. Erdős won the bet but later told Graham that he had set mathematics back a month, and Erdős returned to using stimulants.

Chvátal asserts that the title of Paul Hoffman's 1998 biography *The Man Who Loved Only Numbers* was "a clear libel," as anyone who ever watched Erdős play ping-pong would attest. Three of the book's many photos, which are mostly taken from George Csicsery's film *N is a Number*, show Erdős playing table tennis.

A half-page in the book discusses how almost all mathematicians are the best in their schools but eventually discover they're no longer the smartest kids on the block. However Erdős, although a child prodigy, never suffered this shock and "remained the smartest kid on the block for the rest of his life." In spite of this he never became a snob, and Chvátal points out that Erdős provides an example that cooperation instead of rivalry makes mathematics rewarding and enjoyable, and his attitude made him lots of new friends and was infectious.

4 Opinion

The book does succeed in its main objective of surveying results of Erdős and others who laid the foundations of discrete mathematics. But its subtitle *A Simple Introduction* is only a half-truth as it's not simple. Even the relatively easy inclusion-exclusion principle for sets is only presented algebraically, with no hint of Venn diagrams for the simpler cases of 2 or 3 sets. The book would best serve as a reference book for computer science or mathematics professionals who want a comprehensive, up-to-date survey, and are willing to accept the book's many very detailed arguments. Interestingly, the only mention I spotted of what Erdős called THE BOOK (of perfect proofs of theorems that God maintains) was a reference in an appendix to Martin Aigner and Günter M. Ziegler's book *Proofs from THE BOOK* (Springer, 2014).

On the more positive side, many chapters begin with an enticing problem or theorem. For example, the chapter on Ramsey's Theorem opens with this problem from a Hungarian mathematics competition for high school students in 1947:

> Prove that in any group of six people, either there are three people who know one another or three people who do not know one another.

A version of this problem for six points in space appeared in the 1953 Putnam Mathematical Competition. Another nice feature for readers is that key definitions are often repeated in different chapters as REMINDERS, and they're also collected in Appendix B. The student-friendly twenty-three page Appendix A, "A Few Tricks of the Trade," presents mathematical tools, even with some proofs, involving powerful inequalities, factorials, binomial coefficients, the binomial and the hypergeometric distributions, and the like.

Finally, some nuts and bolts. There's an extensive Bibliography with 378 items, of which at least 55 are by Erdős, but the Index is a bit thin at only 5 pages. The book is attractive, with 4 photos of Erdős on its cover, and easy to lug around, being about 9.5" by 6.5" by 0.5".