Review of[1]

Mathematical Logic: An Introduction

David W. Cunningham

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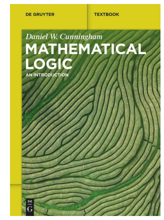

Review by
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One of my colleagues once said to me, "I don't do proof theory. I do mathematics." I guess logic (of the strictly mathematical kind) is one of those subjects that's too easily taken for granted by the working mathematician. Indeed, a firm grounding in mathematical logic does not seem to be regarded as an absolute necessity to the math or CS undergraduate. However, this strikes me as a gap that is worthy of being filled, and the subject provides a perfect landscape for a deep understanding of how proofs work in practice, with all due respect to my colleague. In this way, a textbook on mathematical logic directed towards the undergraduate, such as this one, is a welcome addition.

Full disclosure, although my research for the past 40 years or so has amounted to proving theorems, I never took a course in logic[2] *per se*. But most of what attracted me to computational complexity in the first place was its foundational issues stemming directly from mathematical logic. My teaching of "logic" has only amounted to those fragments of it introduced in discrete math, and those refreshers that one typically gives in algorithms and computability/complexity courses. So seminal results such as Gödel's theorem, for me, are principally viewed from the computational standpoint, and reading through this book was a very enlightening experience.

The present volume starts at the very beginning and over the course of six chapters engages in a rigorous climb, passing many fine views, and at last achieving the summit of Gödel's incompleteness theorems. The contents of the chapters are roughly as follows:

1. *Chapter 1, Basic set theory and basic logic:* This establishes standard basic notations and concepts, such as functions and relations, basic logic and logical connectives, and also including elementary set theory, countablity and cardinality. One distinguishing feature is an explicit theorem that very carefully shows how functions and sets can be defined by recursion. This tool is used throughout the text, for example to formalize languages and (e.g., in Chapter 3) to extend functions on terms to functions on formulas and prove they are unique.

2. *Chapter 2, Propositional logic:* Here the basic logic of Chapter 1 is expanded to more precisely encompass the relevant language of propositional logic, the technical meaning of truth assignments, tautologies, satisfiability, etc. Theorems on tautological completeness and compactness are then proved, and the idea of deducibility introduced.

[2]Having been originally trained as a physicist, this is hardly surprising.

3. *Chapter 3, First-order logic:* Those first two chapters were the foothills, and here the ascent gets steeper as the subject grows in significance. Here we grapple with the meanings of "truth" and "proof." After a substantial section on the language of first-order logic and its structure, we delve into the nature of truth, encapsulated in Tarski's definition of satisfaction. We next encounter structures, definability in a structure, the beginnings of model theory, and the homomorphism theorem of logic. From here we go to the notion of proof, founded on the idea of formal deductions from logical axioms via rules of inference.

4. *Chapter 4, Soundness and completeness:* If something is provable, is it true? Moreover, if something is true, is it provable? This chapter revolves around two key properties of first-order logic: first, the soundness theorem (if a set of wffs is true in a structure, then so is any wff that is deducible from that set); and second, Gödel's *completeness* theorem (given a set of axioms that are true in all models, any statement that is true in those models is deducible (i.e., provable) from those axioms). One consequence of completeness is the Compactness Theorem, which states that if every finite subset of a set Γ of wffs is satisfiable, then Γ is satisfiable. The chapter concludes with applications of the Completeness and Compactness. Consequences of the latter include nonstandard models and Robinson's idea of nonstandard analysis. The basic results underlying nonstandard models of arithmetic are presented. There are also sections presenting the Löwenheim-Skolem theorems, theorems regarding the completeness and axiomatizability of logical theories, and finally proving the correctness of prenex normal form. (At least up to this point, this was my favorite chapter.)

5. *Chapter 5, Computability:* Here we start with a fairly standard informal, intuitive notion of algorithm, which is used to illustrate the notion of decidable sets, total, partial, and computable functions, and the undecidability of the halting problem. There follows a section on formalizations, focussing especially on the Turing machine, and partial recursive functions, as built out of primitive recursion and "partial search" (the unbounded μ-operator), all defined here. It ends with a statement of the Church-Turing Thesis. The next section is a detailed investigation of primitive recursive functions, with a long list of results of such functions and relations. Building on such constructs as bounded quantification and bounded recursion, and thus bounded search and the bounded μ-operator, one can obtain results such as the primitive recursiveness of primality and finding the n^{th} prime number, as well as coding and decoding a number as a sequence of numbers, using the unique factorization into primes. All these in turn lay the groundwork for the primitive recursiveness of Gödel numbers, of central importance in the coming chapter. (This chapter was enjoyable and certainly closest to my own comfort zone.)

6. *Chapter 6, Undecidability and incompleteness:* Much of the preceding five chapters set the stage for this one. It begins by posing the question of whether we can determine if a set of axioms is decidable. The second section introduces a (finite) set of axioms for basic number theory, which incorporates successor, (in)equality, addition, multiplication, and exponentiation, which it calls the Ω-axioms, with the theory thereof denoted $\mathcal{L}$. A number of positive results are then proved to indicate the strengths of the theory, e.g., that Ω can deduce terms involving addition, multiplication, and exponentiation, as well as any true quantifier-free formula and existential sentences. The results are then generalized, showing how $\mathcal{L}$ can represent functions and relations. We can say a formula represents a function if (given the "input" to the function and its "output") it is true iff the function equals its output, i.e., expressed the

graph of the function. Thus we find that we can, in this sense, deduce (from Ω) a formula in $\mathcal{L}$ representing the graph of a function. A series of results builds up to the fact that any recursive function is representable in this language. We then move to the arithmetization of $\mathcal{L}$. Leveraging the results of the previous chapter, a series of results leads to the primitive recursiveness of a number of functions and relations, e.g., the Gödel numbers, determining if (the Gödel number of) one formula is a generalization of another, the set of tautologies, and indeed every representable function and relation. Finally, we proceed to the incompleteness theorems of Gödel, largely on with the fixed-point lemma of logic. It uses this to prove Tarski's Undefinability Theorem, which essentially says that the set of true first-order formulas is not definable within arithmetic. Although Tarski's theorem post-dated the incompleteness theorems, Gödel came upon a form of it while proving them, and it can be used to that purpose, as is done here. From this, Gödel's first incompleteness is quickly stated and proved, and similarly for the second one. (And *this* one was my favorite chapter of all!)

The early sections already display great concern for the reader; some proofs are quite detailed. The same is true of definitions. One instance of this concerns a definition (3.1.6) of terms, as extended from variables and constants, by a set of functions. As it can be a little abstract, the text follows through, in making the definition more concrete by example, in terms of one of the earlier theorems on recursion. Examples (and *non*-examples, something too often missing) are frequently serve to motivate definitions. Many problems are stated and solved in detail in the text itself, which is great preparation for the student working the exercises without the benefit of solutions. Of course, as the book progresses, it leads the reader into deeper territory. As a consequence, more is expected of the reader to fill in the details (indeed some exercises are used in proving later results). Of course this encourages more and more active reading, a rewarding experience which counts as solid pedagogy. Nevertheless, the most important theorems (to cite just one example, the Completeness Theorem of Chapter 4, using Henkin's method of adding witnesses) are carefully and cogently presented.

In conclusion, this book is eminently suited to a junior- or senior-level course, and I learned a great deal from it. It is also worth mentioning another book of the author's on set theory, reviewed by me on these pages[3], which makes an excellent companion volume to this one.

[3] *Set Theory: A First Course*, by Daniel W. Cunningham, *ACM SIGACT News* **48**(3), (2017), pp. 7-9, review by F. Green