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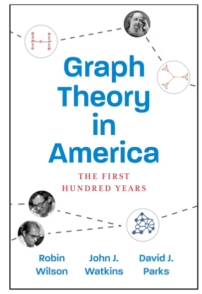

Graph Theory in America:
The First Hundred Years
Robin Wilson, John J. Watkins, and David J. Parks

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Review by
Peter Ross (pross@scu.edu)

1 Introduction

The book focuses on the development of graph theory in the USA and Canada from 1876 to 1976. This is the period from when J. J. Sylvester arrived at Johns Hopkins University from England in 1876, until the proof of the Four Color Theorem by Appel and Haken at the University of Illinois in 1976. But the book covers much more, including a preliminary chapter on early American mathematics, two *Interludes* that are essentially short chapters on graph theory in Europe, and an *Aftermath*, a eight-page summary of progress in graph theory in recent years.

Graph Theory in America (GTA) is based on a doctoral dissertation by co-author David Parks under the supervision of co-author Robin Wilson. Its Preface states, “No prior knowledge of graph theory is required in reading this book, which aims to explain the historical development of the subject in simple terms to a general reader interested in mathematics.” It adds that “readers who are interested mainly in the historical narrative, and in the personalities involved, can safely pass over any technical material.” I believe that GTA has done this successfully, and my review will try to focus equally on the history and the mathematics involved.

2 Summary

The six main chapters cover chronologically the 1800s through the 1960s and 1970s. Each chapter discusses advances in graph theory and the key mathematicians involved, as well as progress on the four color problem (FCP) and eventually the Four Color Theorem. A preliminary chapter, “Setting the Scene,” focuses on Benjamin Peirce at Harvard and especially Eliakim Hastings Moore at the University of Chicago, who was there from 1892 until his death in 1932. Moore worked in algebra, the foundations of geometry, and analysis, but never in graph theory. However, two of his 31 doctoral students, Oswald Veblen and George Birkhoff, made significant contributions to graph theory that are discussed in Chapter 2 on the 1900s and 1910s.

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The story of the FCP begins in Chapter 1, on the 1800s, when Francis Guthrie, a former student of De Morgan, stated it in 1852. The problem intrigued Hamilton and also Cayley, who showed in 1878 that to prove that four colors can color any (planar) map, it suffices to show that they can color any cubic map, one with 3 countries at each meeting point or vertex. In 1879 Cayley's former student Alfred Bray Kempe published a purported solution to the FCP in the new *American Journal of Mathematics*, whose editor in chief was J. J. Sylvester, a friend of Cayley. Kempe's "proof" was generally accepted until 1890, when Percy Heawood exposed its fatal error. Much of Chapter 1 discusses Sylvester's fascinating career, including his back-and-forth moves to England and his innovative uses of graphs to represent molecules in chemistry and "binary quantics" in algebra, and an analogy between them. The book mentions that his 1873 note in *Nature* on this analogy included the first use of the word "graph" in our modern sense.

The first *Interlude* on graph theory in Europe discusses Heawood's ground-breaking paper "Map-colour Theorem". The paper not only exposed the fatal flaw in Kempe's long accepted "proof" but also showed that 5 colors suffice for a map on a plane or sphere by modifying Kempe's arguments. Heawood went on to study coloring of other orientable surfaces like a torus. He proved that the chromatic number of a torus, the smallest number of colors to color every map on it, is 7. He further conjectured the general formula $\lfloor\frac{7+\sqrt{1+48g}}{2}\rfloor$ for the chromatic number of an orientable surface of genus g (g handles or holes) where $g \geq 1$. This so-called Heawood Conjecture was not proven for over seventy years, until Gerhard Ringel and Ted Youngs proved it at the University of California at Santa Cruz in 1968. The *Interlude* chapter discusses other notable European mathematicians like Julius Petersen of Denmark, for whom the Petersen graph is named, Heinrich Tietze of Austria, who studied the chromatic number of non-orientable surfaces, and Hermann Minkowski of Germany. It also repeats an amusing story from Constance Reid's biography of Hilbert, of how Minkowski interrupted his topology lecture at Göttingen to assert that the Four Color Theorem "has not yet been proved, but that is because only mathematicians of the third rank have occupied themselves with it." Minkowski added that he believed he could prove it, and proceeded to work on it in class for several weeks, before giving up and admitting his arrogance.

Chapter 2 on the 1900s and 1910s discusses significant contributions to graph theory by George D. Birkhoff at Harvard and Oswald Veblen at Princeton. Veblen used algebra to discuss maps, like modular equations and incidence matrices for graphs, following Henri Poincaré. Veblen's influential 1922 monograph *Analysis Situs* on combinatorial topology was based on his 1916 *AMS Colloquium Lectures in Cambridge, MA*. Among other things the lectures introduced the concepts of rank and nullity of a map or planar graph. They also developed some 19th century ideas of G.R. Kirchhoff who, in studying charges in electrical networks, introduced the idea of a spanning tree. George D. Birkhoff, "the leading mathematician of his time," had a mild obsession with the FCP according to the GTA book, but in later life Birkhoff rued the time and effort he spent on it. However, Birkhoff once told Hassler Whitney that "every great mathematician (of the time) had studied the problem, and thought at some time that he had proved the theorem." Birkhoff's first paper on the subject introduced the chromatic polynomial of a map $P(\lambda)$, which counts the possible colorings of the map with λ colors. $P(\lambda)$ is of degree n, where n is the number of regions of the map. His main objective was to prove $P(4) > 0$, the Four Color Theorem, but his methods involving determinants were cumbersome. However, twenty years later Whitney discovered a simpler procedure for determining the coefficients of $P(\lambda)$. Birkhoff was regarded as North America's leading mathematician after he solved a restricted form of the three-body problem, like finding the motion of the sun, Earth, and moon, which Poincaré had been unable to solve.

Chapter 3 on the 1920s focuses a lot on Philip Franklin, whose doctoral thesis, supervised by Veblen, was on the FCP. It recounts an amusing story by David Widder, who had a bunk in the same barracks as fellow mathematicians Franklin and Norbert Wiener during World War I, while working at the Aberdeen Proving Ground in Maryland. Franklin and Wiener sometimes talked mathematics so far into the night that, in order to get some sleep, Widder once hid the light bulb! In later life Franklin and Wiener became brothers-in-law at MIT, where both spent the majority of their careers, and where each brought different forms of topology to the institute. Franklin's 1922 paper on the FCP proved that the conjecture was true for every map on the plane (or sphere) with 25 or fewer regions. In the late 1930s Franklin improved this to 31 or fewer regions, followed by C.E. Winn, who showed 35 or fewer regions sufficed, a result that held for thirty years. Also in the 1930s Franklin finished off a Heawood conjecture for non-orientable surfaces of genus g, where g is the number of cross-caps added to a sphere, so g is 1 for the projective plane and 2 for the Klein bottle. Work by H. Tietze in 1910 had led to a Heawood conjecture for non-orientable surfaces of genus $g \geq 1$, that their chromatic number was $\lfloor \frac{7+\sqrt{1+24g}}{2} \rfloor$. Franklin proved that this failed for $g = 2$, and in 1952 Gerhard Ringel showed that Franklin had found the only instance where Tietze's formula fails.

A second *Interlude* on graph theory in Europe discusses the work of Polish topologist Kazimierz Kuratowski on planar graphs. His central theorem of 1922 asserted that a graph is planar if and only if it has no subgraph that is homeomorphic to the complete graph K_5 or the complete bipartite graph $K_{3,3}$. Kuratowski had "a most distinguished career," collaborating with major figures like Stefan Banach, Max Zorn (of Zorn's Lemma), and John von Neumann, and he served as "a world ambassador for Polish mathematics."

Chapter 4 on the 1930s sees the prestige of graph theory so low that one mathematician then called it "the slums of topology." Two important American mathematicians, Hassler Whitney and Saunders Mac Lane, changed this. In the early 1930s Whitney wrote a dozen papers on major areas of graph theory like coloring, planarity, duality, and matroids. Duality included the combinatorial dual of a graph, which is sometimes called the Whitney dual. Matroids arose from Whitney noticing similarities between the concepts of rank and independence in graph theory and those of dimension and linear independence for vector spaces. GTA briefly surveys Whitney's major contributions to other topics than graph theory, such as his strong embedding theorem for n-dimensional differentiable manifolds, the cup product in cohomology rings, and work on singular spaces and the singularities of smooth maps, which eventually led to catastrophe theory and chaos theory. Saunders Mac Lane met Whitney while both were young faculty members at Harvard in the mid-30s. Mac Lane extended Whitney's work on combinatorial graphs in three papers, before moving on to other fields like algebraic topology and category theory, where he had a significant impact. His texts also were a major influence on mathematics education in universities. Along with Ron Graham, Mac Lane was President at times of both the AMS and the MAA.

Chapter 4 ends with 8 pages on academic life in the 1930s, discussing the effects of the Great Depression and the early years of the war in Europe. This included the birth of the AMS's *Mathematical Reviews*, which was proposed by Veblen after G.H. Hardy, Harald Bohr and others resigned from the reviewing journal *Zentralblatt für Mathematik* due to its actions that were influenced by the Nazi Party. Another controversy, which 8 pages deal with, is that of antisemitism in some mathematics departments against Jews, especially refugees from the Nazis, who were viewed as taking jobs away from budding American mathematicians. A prominent leader with this opinion was George Birkhoff at Harvard, who in 1934 even initially opposed Solomon Lefschetz's becoming the first Jewish president of the AMS.

The meaty fifty-page Chapter 5 on the 1940s and 1950s focuses on further contributions to graph theory by established mathematicians like Birkhoff, and new contributors like Canadian Bill Tutte and Americans Claude Shannon and Frank Harary. Bill Tutte left Cambridge University in January 1941 to join Alan Turing at the highly secret Bletchley Park near London working as a codebreaker. The book says that his work in unraveling the internal workings of the German cipher machine that replaced the Enigma machines was "an astonishing feat of cryptanalysis that is sometimes believed to have shortened the war by two years or more." After the war Tutte returned to Cambridge, earning his doctorate with a 417-page dissertation *An Algebraic Theory of Graphs*, which he later said attempted to reduce graph theory to linear algebra. GTA points out that the thesis included the first major advances on matroids since their introduction by Whitney in 1935. After getting his doctorate, Tutte joined H.S.M. (Donald) Coxeter at the University of Toronto for 14 years, before moving to the University of Waterloo that's also in Ontario, for the last 36 years of his long career. GTA devotes 11 pages to some of Tutte's work, including showing his example of a cubic polyhedron (so of degree 3) with no Hamiltonian cycle, thus disproving Tait's 1884 claim that every cubic polyhedron has one. The book also discusses Tutte's discovery of what became known as the Tutte polynomial, a polynomial in two variables, and his later work at Waterloo on chromatic polynomials.

Chapter 5 includes a concise 11-page summary of progress on algorithms, which arose from practical problems in World War II and later from the budding computer industry. These algorithms include ones for matching and assignment, transportation, linear programming, flows in networks, finding minimum spanning trees, search algorithms, and path problems such as finding the shortest or longest paths in graphs. The final pages of the chapter are devoted to Frank Harary, who came to be known as "the father of modern graph theory." Harary wrote eight books and more than seven hundred papers in his long career at Michigan and later New Mexico State University in Las Cruces. GTA discusses just three of the many areas that Harary worked in: signed graphs, graph enumeration that was based on earlier work by Cayley and Polya, and Ramsey graph theory. Ramsey theory became popular, according to the GTA, after a coloring problem for the complete graph K_6 appeared on the 1953 Putnam exam that was proposed by Harary. Harary may have gotten the problem from this similar one from the Hungarian Mathematical Olympiad of 1947: *If there are six people at a party, prove that there must be at least three mutual friends or three mutual non-friends.* GTA presents an elegant two-sentence solution to the graph theory version of the problem.

Chapter 6, the 1960s and the 1970, opens by observing that graph theory was increasingly becoming part of mainstream mathematics, and the two decades saw the long-awaited proofs of both the Heawood conjecture and the Four Color Theorem. Oystein Ore, a Norwegian whose long career was at Yale, wrote three books on graph theory. The first, his 1962 *AMS Colloquium Lecture series on the Theory of Graphs*, was one of the first two texts in English on the subject. Ore along with two doctoral students, well-known combinatorialist Marshall Hall, Jr., and Joel G. Stemple, showed that a planar map not colorable in four colors must have at least 40 countries. Others extended this result to 48 and then 96 countries, but the GTA authors mischievously note, "there was still a long way to go." As noted earlier, the Heawood formula for the chromatic number of any orientable surface of genus g was finally proven by Ringel and Youngs in 1968. The book examines some of the cases and the ideas in their proof, which it calls a *tour de force*, and notes that the theorem is now known as the Ringel-Youngs theorem. Surprisingly, the Heawood conjecture for non-orientable surfaces had been completely settled by Ringel earlier in 1954, and GTA briefly discusses why those surfaces were easier to deal with. Gerhard Ringel had a very unusual background for a mathematician. Raised in Czechoslovakia and graduating from Charles University in Prague, he was drafted into the Wehrmacht in World War II and later spent four years as a POW in a Soviet jail. After being released he obtained a doctoral degree from the University of Bonn in 1951. He taught for many years in Germany, then moved to the University of California in Santa Cruz, where he succeeded his friend Ted Youngs who had retired.

Another "colorful character" who's described in Chapter 6 is Ron Graham, whom G.-C. Rota called "the leading problem-solver of his generation" in his 1991 nomination of Graham for AMS President. Graham never graduated from high school, due to his father changing jobs around the country. He funded his doctoral studies at Berkeley in the early 1960s with a trampolining troupe and a juggling act that he formed and performed in circuses and elsewhere. Persi Diaconis once said, "Ron, as much as anybody, is responsible for bringing high-powered math to bear on computer science." GTA discusses in detail one of Graham's approximately 400 papers, that was written with his Bell Labs colleague Henry O. Pollak, and that was motivated by telephone switching theory. Graham and Pollak introduced the distance matrix of a graph, whose (i, j) entry was the length of the shortest path between vertices i and j. Pollak later recalled that people had previously only studied the adjacency matrices of graphs. Graham and Pollak's main theorem had wide-ranging consequences, including a simple formula for the determinant of the distance matrix of a tree, which turned out to be independent of the structure of the tree. GTA points out that a clever proof of the formula is based on a determinant rule discovered by the 19th century English mathematician C.L. Dodgson, better known as writer Lewis Carroll. GTA briefly discusses Graham's extensive writings on Ramsey theory, many with his wife Fan Chung, and his very unusual friendship and collaboration with, and stewardship of, Paul Erdős. Graham even defined the amusing Erdős number of an author, as the length of the shortest chain of collaborators from Erdős to the author.

Chapter 6 includes an excellent six-page section on Complexity, which features important contributions by Jack Edmonds, who may have originated the fundamental question *Is P = NP?* in 1967, and Stephen Cook's definition of NP-complete problems, along with some of his startling results. GTA mentions that despite a Clay Mathematics Institute prize of one million dollars for deciding if P = NP, "since the 1970s, little progress has been made in settling this general problem." I was surprised in reading the concise section on Complexity to find out that John von Neumann caused interest to increase in the efficiency of algorithms, when in 1953 he apparently was the first to distinguish between polynomial-time and exponential-time algorithms in an article linking game theory to the optimal assignment problem.

The final sentence of Chapter 6, at the end of 18 pages of progress on the FCP by Wolfgang Haken, Heinrich Heesch, Kenneth Appel and others, simply states **the Four Color Theorem was proved**. Haken made a name for himself in the early 1950s by solving a long-standing unsolved knot problem, determining if a given tangle of string contains a knot. This led to his giving an invited lecture at the 1954 International Congress of Mathematicians in Amsterdam. Afterwards he moved from the University of Kiel in Germany to the University of Illinois, where in 1967 he got four color problem expert Heesch to visit. Heesch had used a CDC 1604A computer for the FCP in Germany, and together Appel and Heesch got time on the Cray Control Data 660 computer, the most powerful machine of the day, at the Atomic Energy Commission's Brookhaven National Laboratory on Long Island. Coincidentally the computer center's director Yoshio Shimamoto was a FCP enthusiast who himself discovered an apparently key configuration of map regions in 1971, which became known as the Shimamoto horseshoe. With help from the "most distinguished graph theorists of the day" Hassler Whitney and Bill Tutte, Shimamoto's approach was found to lead to a dead end. By the early 1970s Haken considered giving up on the FTP, and in a lecture in Illinois he described computer experts' negative views on the massive calculations that his research plan needed. Fortunately Kenneth Appel, a mathematician in the audience, had extensive programming experience and differed with the experts' opinions and volunteered to help. Appel and Haken started working together in 1972, but it took four more years, with a new approach and a powerful new computer at Illinois that only Appel seemed to be able to get to run properly, before in late June 1976 Appel placed a notice on the department's blackboard saying, "Modulo careful checking, it appears that four colors suffice." The phrase *four colors suffice* became the department's postal meter slogan. At the same time, others using similar methods at the University of Waterloo, the University of Rhodesia, and Harvard expected success within months. Appel and Haken, using their children to help check the final configurations, went public on July 22, 1976. A day later *The Times* of London reported their proof, adding that it contained 10,000 diagrams and that the computer printout stood four feet high on the floor.

An eight-page *Aftermath* chapter in GTA briefly summarizes the huge increase in graph theory and combinatorics in America since the proof. It lists fourteen research topics, about half of which are new to the book. The *Aftermath* ends by asserting "The development of graph theory in America over the century from 1876 to 1976 was truly remarkable ...," listing ten mathematicians who were undoubtedly among the most significant. Each of these has been mentioned in this review.

3 Opinion

It's probably clear by now that I think very highly of the book. It will likely interest readers who don't know graph theory but do know some history of mathematics, and those who don't know much history of mathematics but do know some graph theory. The authors are good story tellers who make graph theory and history come alive. I noticed no technical errors, and in fact spotted not a single misspelling or typo.

The book has many nice features:

- short summaries of some influential papers that influenced the subject's development;
- a paragraph or two at the beginning or end of each chapter, giving a preview of what's coming and serving as an advanced organizer;
- a ten-page glossary of technical terms and a five-page index;
- twenty pages of detailed notes, references, and further reading;
- a nine-page chronology of events, beginning with the founding of Harvard in 1636 and ending with the awarding of the Abel Prize to László Lovász and Avi Wigderson in 2021;
- photos, many just head shots, of almost all of the major mathematicians discussed in the book.

Finally, the hardcover book is well-made, surviving my extensive annotations, and is very reasonably priced, available from Amazon for less than $25.