Review of[1]
Combinatorial Game Theory
A Special Collection in Honor of Elwyn Berlekamp, John H. Conway and Richard K. Guy
Eds. R. J. Nowakowski, B. M. Landman, F. Luca, M. B. Nathanson, J. Nešetřil, and A. Robertson
de Gruyter, 2022
413 pages

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1 Introduction

This book is devoted to the remembrance of the work of the three mathematicians: Elwyn Berlekamp, John Conway, and Richard Guy. It includes twenty writings from researchers that reflect on their work in combinatorial game theory. The book is available in eBook and hardcover editions and is priced at 230 US dollars. The ISBNs of the eBook and hardcover editions are 9783110755411 and 9783110755343 respectively. The book includes contributions by leading experts in combinatorial game theory. It notes the work done by Berlekamp, Conway and Guy. The book has been published as part of the De Gruyter Proceedings in Mathematics. It should be of interest to researchers and students working in combinatorial game theory.

2 Summary

The book comprises twenty chapters. The first chapter is on the game of Flipping Coins. It is based on the game TURNING TURTLES introduced by Berlekamp, Conway, and Guy. The TURNING TURTLES game consists of a finite row of turtles. These turtles may either be on their feet or may lie on their backs. In order to make a move, a player has to turn one turtle over onto its back, with other options being available. Coins may be used instead of turtles. The authors of this chapter focus partly on winning strategies. One of their key results is that all the Flipping Coins positions are numbers. They also look very briefly at other problems worth investigating.

The second chapter is on the game of Blocking Pebbles. This chapter conforms graph pebbling, a well-known single-player game on graphs into a two-player strategy game. A two-player combinatorial ruleset based on graph pebbling is studied along with some families of game values that are accomplishable in blocking pebbles. The game of blocking Pebbles when restricted to green pebbles is analyzed and the computational complexity of blue-red-green blocking pebbles is determined as being PSPACE-hard.

The third chapter on *Transverse Wave: an impartial color-propagation game inspired by social influence and Quantum Nim* studies an impartial combinatorial game played on a 2D grid. The authors like this game due to its seeming ease, intractableness, and underlying association to two other combinatorial games: one about social influences and another on quantum superpositions. They also obtain complexity-theoretic results.

The chapter *A note on numbers* considers the often asked question "*When are all positions of a game numbers?*". The authors show that two properties known as F1 and F2 are necessary and sufficient. These properties are results of the fact that, in a number, it is not a benefit to be the first player. Some illustrative examples are given and a warning is also included.

The chapter on *Ordinal sums, CLOCKWISE HACKENBUSH, and DOMINO SHAVE* presents two rulesets: CLOCKWISE HACKENBUSH and DOMINO SHAVE. CLOCKWISE HACKENBUSH is stated to be a new variant of hackenbush trees, while DOMINO SHAVE is the partizan version of Stirling Shave. The authors remark that DOMINO SHAVE is in someways normal and has, as special cases, Stirling Shave and Hetyei's Bernoulli game. They prove that although CLOCKWISE HACKENBUSH seems artificial, yet it is equivalent to DOMINO SHAVE. The authors generalize van Roode's signed binary number method for blue-red hackenbush.

Poset games are impartial combinatorial games whose game boards are partially ordered sets (posets). The chapter *Advances in finding ideal play on poset games* demonstrates systematic ways to compute the *nimber* (also called Grundy number) of poset games while permitting the categorization of winning or losing positions. The outcomes exhibit a comparison of idealistic strategies on posets that are apparently unconnected. The authors focus on normal play games, where the first player having no move loses. They also study equality of games and some of its applications.

The chapter on *Strings-and-Coins and Nimstring are PSPACE-complete* obtains complexity-theoretic results related to two games viz. Strings-and-Coins and Nimstring. In fact, the authors show that Strings-and-Coins, a combinatorial two-player game which is a variant of the Dots-and-Boxes game (a simple pencil-and-paper game on a grid of dots) considered by Berlekamp is indeed strongly PSPACE-complete on multi-graphs. This betters the best previous record, NP-hardness mentioned in the book Winning Ways by Berlekamp, Conway, and Guy. Finally, the authors pose some open problems.

Partizan subtraction games are two-player combinatorial games played on a heap of tokens. Each player is assigned a finite set of integers. A move consists in removing a number m of tokens from the heap, provided that m belongs to the set of the player. The first player unable to move loses. Fraenkel and Kotzig discussed these games and they introduced the notion of dominance. In this chapter, the authors investigate other kinds of behaviors for the outcome sequence. In addition to dominance, three other disjoint behaviors are defined: weak dominance, fairness, and ultimate impartiality. The authors also study complexity and obtain several other results.

The chapter on *Circular Nim games CN(7, 4)* looks at Circular Nim. Circular Nim is a two-player impartial combinatorial game comprising n stacks of tokens set in a circle. A move consists of selecting k successive stacks and choosing at least one token from one or more of the stacks. The last player able to make a move wins. The question investigated by the authors is: *Who can win from a given position if both players play optimally?* The authors obtain results for $n = 7$ and $k = 4$.

Misère domineering on 2 × n boards studies domineering - a tiling game, in which one player puts vertical dominoes, and a second puts horizontal dominoes, both flip-flopping turns until somebody cannot put, on their turn. The authors state that past research has found game outcomes and values for some rectangular boards under normal play (last move wins); however, nothing was known about domineering under misère play (last move loses). The authors look at Misère outcomes of 2 × n domineering and the algebra of misère domineering. They study optimal-play outcomes for all 2 × n boards under misère play. Misère outcomes for m × n domineering are also shown.

The chapter on *Relator games on groups* specifies two impartial games, the Relator Achievement Game REL and the Relator Avoidance Game RAV. When presented a finite group G and a generating set S, both games commence with the empty word. Two players make a word in S by having turns adding an element from $S \cup S^{-1}$ at each turn. The first player to make a word equivalent in G to a former word wins the game REL but loses the game RAV. We can look upon REL and RAV as *make a cycle* and *avoid a cycle* games on the Cayley graph $\Gamma(G, S)$. The authors determine winning strategies for various families of finite groups including dihedral, dicyclic, and products of cyclic groups. Three player games and open questions are also discussed.

Playing Bynum's game cautiously examines a version of Bynum's game, also known as Eatcake. Various sequences are brought in with the intent of analyzing Eatcake. Two of these have terms with uptimal values (notions due to Conway and Ryba, the 1970s). All others (eight) are determined by "uptimal+ forms," i.e., standard uptimals plus a fractional uptimal. The game itself is played on an n × m grid of unit squares. The author describes all sub-matrices of the 12 × 12 grid. Values of larger positions are also investigated.

Genetically modified games looks at the application of genetic programming to games. Genetic programming is often considered as a technique of evolving programs, starting from a population of unfit programs, fit for a particular job by applying operations similar to natural genetic processes to the population of programs. It uses crossover and mutation of genes representing functional operations. The authors acquaint and solve two combinatorial games, and present some advantages and disadvantages of using genetic programming. They study a combinatorial game whose ruleset and starting positions are helped by genetic structures. A single-point crossover and mutation game and a two-point crossover game are studied along with a crossover-mutation game.

Game values of arithmetic functions investigates two-player games inspired by standard arithmetic functions, such as Euclidian division, divisors, remainders, and relatively prime numbers. Games based on the aliquots, aliquants, and totatives, counting games, dividing games, factoring games, full set games and powerset games are discussed.

The next chapter is on *a base-p Sprague–Grundy-type theorem for p-calm subtraction games*. Base 2 arithmetic has a central function in combinatorial game theory. Nevertheless, a few games related to base p have also been found, where p is an integer greater than 1, although not necessarily a prime number. The author introduces a notion called as *p-calm* and points out that Nim and Welter's game are p-calm. Furthermore, using the p-calmness of Welter's game, the author extrapolates a connection between Welter's game and representations of symmetric groups to disjunctive sums of Welter's games and representations of generalized symmetric groups. Subtraction games and p-calm subtraction games are discussed.

The chapter on *recursive comparison tests for dicot and dead-ending games under misère play* applies the theory of absolute combinatorial games to formulate recursive comparison tests for the universes of dicots and dead-ending games. This is claimed by the authors as the first constructive test for comparability of dead-ending games under misère play using a novel category of end-games called *perfect murders*.

The chapter on *impartial games with entailing moves* specifies a notion called as *affine impartial*, which broadens typical impartial games. It analyzes their algebra by widening the conventional Sprague–Grundy theory with an ensuant minimum excluded rule. Results of NIMSTRING and TOP ENTAILS are given to exemplify the theory.

The next chapter is on *Extended Sprague–Grundy theory for locally finite games, and applications to random game-trees*. The Sprague–Grundy theory for finite games without cycles was widened to general finite games in earlier works. The authors of this chapter state that the framework used to sort out finite games also extends the case of locally finite games (that is, games where any position has only finitely many options). Especially, any locally finite game is tantamount to some finite game. The authors then examine cases where the directed graph of a game is selected randomly and is given by the tree of a Galton–Watson branching process. The authors study Extended Sprague–Grundy theory for games with infinite paths, reduced graphs, random game trees, and also provide examples.

The next chapter is on *Grundy numbers of impartial three-dimensional chocolate-bar games*. Chocolate-bar games are essentially versions of the Chomp game. A two-dimensional chocolate bar is a rectangular array of squares in which some squares are removed throughout the course of the game. A "poisoned" or "bitter" square, typically printed in black, is included in some part of the bar. A three-dimensional chocolate bar is a three-dimensional array of cubes in which a poisoned cube printed in black is included in some part of the bar. A three-dimensional chocolate bar is essentially composed of a set of $1 \times 1 \times 1$ cubes with a "bitter" or "poison" cube at the bottom of the column at position (0, 0). Two players get turns to prune the bar along a plane horizontally or vertically along the grooves, and eat the exposed pieces. The player who is fruitful to pass on the opponent with a single bitter cube is the winner. The authors determine Grundy numbers of impartial three-dimensional chocolate-bar games and also look at unsolved problems.

On the structure of misère impartial games studies the abstract structure of the monoid M of misère impartial game values. The author begins with the prerequisites and then demonstrates various new results, including a proof that the group of fractions of M is almost torsion-free. In addition, a method for estimating the number of distinct games born by day n, and some new results on the structure of prime games are illustrated.

3 Opinion

This book is indeed a special and well-prepared collection in honor of Elwyn Berlekamp, John H. Conway and Richard K. Guy, the mathematicians who contributed immensely to combinatorial game theory. The book is worth reading several times for those interested in game theory. It is a tribute to the work of the legends.